[C100/SQP255]

Mathematics
Advanced Higher

Time: 3 hours

NATIONAL QUALIFICATIONS

Specimen Question Paper
for use in and after 2004

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions .
3. **Full credit will be given only where the solution contains appropriate working.**

Answer all the questions.

Marks

1. (*a*) Find partial fractions for

$$\frac{4}{x^2 - 4}.$$ **2**

(*b*) By using (*a*) obtain

$$\int \frac{x^2}{x^2 - 4}\,dx.$$ **4**

2. Use the Euclidean Algorithm to find integers of x, y such that

$$195x + 239y = 1.$$ **5**

3. The performance of a prototype surface-to-air missile was measured on a horizontal test bed at the firing range and it was found that, until its fuel was exhausted, its acceleration (measured in $\mathrm{m\,s^{-2}}$) t seconds after firing was given by

$$a = 8 + 10t - \frac{3}{4}t^2.$$

(*a*) Obtain a formula for its speed, t seconds after firing. **2**

(*b*) The missile contained enough fuel for 10 seconds. What horizontal distance would it have covered on the firing range when its fuel was exhausted? **3**

4. The $n \times n$ matrix A satisfies the equation

$$A^2 = 5A + 3I$$

where I is the $n \times n$ identity matrix.

Show that A is invertible and express A^{-1} in the form of $pA + qI$. **2**

Obtain a similar expression for A^4. **2**

5. Use the substitution $x = 4 \sin t$ to evaluate the definite integral

$$\int_0^2 \frac{x+1}{\sqrt{16 - x^2}}\,dx.$$ **5**

Marks

6. Use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= -1 \cdot 1 \\ x + 3y + 2z &= 0 \cdot 9. \end{aligned}$$ **5**

7. Use Maclaurin's theorem to write down the expansions, as far as the term in x^3, of

(i) $\sqrt{1+x}$, where $-1 < x < 1$, and **3**

(ii) $(1-x)^{-2}$, where $-1 < x < 1$. **2**

8. (*a*) Find the derivative of y with respect to x, where y is defined as an implicit function of x by the equation

$$x^2 + xy + y^2 = 1.$$ **2**

(*b*) A curve is defined by the parametric equations

$$x = 2t + 1, \qquad y = 2t(t - 1).$$

(i) Find $\frac{dy}{dx}$ in terms of t. **2**

(ii) Eliminate t to find y in terms of x. **1**

9. Let $u_1, u_2, \ldots, u_n, \ldots$ be an arithmetic sequence and $v_1, v_2, \ldots, v_n, \ldots$ be a geometric sequence. The first terms u_1 and v_1 are both equal to 45, and the third terms u_3 and v_3 are both equal to 5.

(*a*) Find u_{11}. **2**

(*b*) Given that $v_1, v_2, \ldots$ is a sequence of **positive** numbers, calculate $\sum_{n=1}^{\infty} v_n$. **3**

10. Use induction to prove that

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

for all positive integers n. **5**

11. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = f(x)$$

in each of the cases

(i) $f(x) = 20\cos x$ **3**

(ii) $f(x) = 20\sin x$ **3**

(iii) $f(x) = 20\cos x + 20\sin x$. **1**

12. Let the function f be given by

$$f(x) = \frac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2}, \quad x \neq 2.$$

(*a*) The graph of $y = f(x)$ crosses the y-axis at $(0, a)$. State the value of a. **1**

(*b*) For the graph of $f(x)$

(i) write down the equation of the vertical asymptote, **1**

(ii) show algebraically that there is a non-vertical asymptote and state its equation. **3**

(*c*) Find the coordinates and nature of the stationary point of $f(x)$. **4**

(*d*) Show that $f(x) = 0$ has a root in the interval $-2 < x < 0$. **1**

(*e*) Sketch the graph of $y = f(x)$. (You must include on your sketch the results obtained in the first four parts of this question.) **2**

13. (*a*) Show that the lines

$$L_1 : \frac{x-3}{2} = \frac{y+1}{3} = \frac{z-6}{1}$$

$$L_2 : \frac{x-3}{-1} = \frac{y-6}{2} = \frac{z-11}{2}$$

intersect, and find the point of intersection. **6**

(*b*) Let A, B, C be the points $(2, 1, 0)$, $(3, 3, -1)$, $(5, 0, 2)$ respectively.

Find $\vec{AB} \times \vec{AC}$.

Hence, or otherwise, obtain the equation of the plane containing A, B and C. **5**

14. Let $z = \cos\theta + i\sin\theta$.

Marks

(a) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ. **5**

(b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ. **1**

(c) Use your answers to (a) and (b) to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. **1**

(d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$ where k, m, n, p are integers. State the values of k, m, n, p. **4**

15. In a chemical reaction, two substances X and Y combine to form a third substance Z. Let $Q(t)$ denote the number of grams of Z formed t minutes after the reaction begins. The rate at which $Q(t)$ varies is governed by the differential equation

$$\frac{dQ}{dt} = \frac{(30-Q)(15-Q)}{900}.$$

(a) Express $\dfrac{900}{(30-Q)(15-Q)}$ in partial fractions. **2**

(b) Use your answer to (a) to show that the general solution of the differential equation can be written in the form

$$A\ln\left(\frac{30-Q}{15-Q}\right) = t + C,$$

where A and C are constants.

State the value of A and, given that $Q(0) = 0$, find the value of C. **4**

Find, correct to two decimal places,

(i) the time taken to form 5 grams of Z, **1**

(ii) the number of grams of Z formed 45 minutes after the reaction begins. **2**

[*END OF SPECIMEN QUESTION PAPER*]